

q -Difference raising operators for Macdonald polynomials and the integrality of transition coefficients

ANATOL N. KIRILLOV^{*1} and MASATOSHI NOUMI^{*2}

Introduction.

The purpose of this paper is study certain q -difference raising operators for Macdonald polynomials (of type A_{n-1}) which are originated from the q -difference - reflection operators introduced in our previous paper [KN]. These operators can be regarded as a q -difference version of the raising operators for Jack polynomials introduced by L. Lapointe and L. Vinet [LV1, LV2]. As an application of our q -difference raising operators, we will give a proof of the integrality of the double Kostka coefficients which had been conjectured by I.G. Macdonald [Ma], Chapter VI. We will also determine their *quasi-classical limits*, which give rise to (differential) raising operators for Jack polynomials. (See also *Notes* at the end of Introduction.)

Let $\mathbb{K} = \mathbb{Q}(q, t)$ be the field of rational functions in two indeterminates (q, t) and $\mathbb{K}[x]^W$ the algebra of symmetric polynomials in n variables $x = (x_1, \dots, x_n)$ over \mathbb{K} , W being the symmetric group \mathfrak{S}_n of degree n . The *Macdonald polynomials* $P_\lambda(x; q, t)$ are a family of symmetric polynomials parametrized by partitions, and they form a \mathbb{K} -basis of $\mathbb{K}[x]^W$. They are characterized as the joint eigenfunctions in $\mathbb{K}[x]^W$ for Macdonald's commuting family of q -difference operators

$$(1) \quad D_r = t^{\binom{r}{2}} \sum_{\substack{I \subset [1, n] \\ |I|=r}} \prod_{\substack{i \in I \\ j \notin I}} \frac{tx_i - x_j}{x_i - x_j} \prod_{i \in I} T_{q, x_i} \quad (r = 0, 1, \dots, n).$$

For each partition λ , the Macdonald polynomial $P_\lambda(x; q, t)$ is the unique joint eigenfunction of D_r ($r = 0, 1, \dots, n$) that has the leading term $m_\lambda(x)$ under the dominance order of partitions when it is expressed as a linear combination of monomial symmetric functions $m_\mu(x)$. As in [Ma] (VI.8.3), we also use another normalization $J_\lambda(x; q, t) = c_\lambda(q, t)P_\lambda(x; q, t)$, called the “integral form” of $P_\lambda(x; q, t)$.

We now define the two kinds of q -difference operators K_m^+ and K_m^- ($m =$

^{*1}Department of Mathematical Sciences, University of Tokyo *and*
St. Petersburg Branch of the Steklov Mathematical Institute

^{*2}Department of Mathematics, Kobe University

$0, 1, \dots, n$) as follows:

$$(2) \quad K_m^+ = \sum_{\substack{J \subset [1, n] \\ |J|=m}} \prod_{j \in J} x_j \sum_{I \subset J} (-t^{m-n+1})^{|I|} t^{\binom{|I|}{2}} \prod_{\substack{i \in I \\ j \in [1, n] \setminus I}} \frac{tx_i - x_j}{x_i - x_j} \prod_{i \in I} T_{q, x_i},$$

$$K_m^- = \sum_{\substack{J \subset [1, n] \\ |J|=m}} \prod_{j \in J} x_j \sum_{I \subset J} (-t)^{m-|I|} t^{\binom{m-|I|}{2}} \prod_{\substack{i \in I \\ j \in [1, n] \setminus I}} \frac{x_i - tx_j}{x_i - x_j} \prod_{j \in [1, n] \setminus I} T_{q, x_j}.$$

These operators are W -invariant and preserve the ring of symmetric polynomials $\mathbb{K}[x]^W$.

Theorem A. *For each $m = 0, 1, \dots, n$, the q -difference operator $K_m = K_m^+$ (resp. K_m^-) is a raising operator for Macdonald polynomials $J_\lambda(x; q, t)$ in the sense that*

$$(3) \quad K_m J_\lambda(x; q, t) = J_{\lambda + (1^m)}(x; q, t)$$

for any partition λ with $\ell(\lambda) \leq m$.

(See Theorem 2.2 in Section 2.)

Theorem A implies that, for any partition $\lambda = (\lambda_1, \dots, \lambda_n)$, the Macdonald polynomial $J_\lambda(x; q, t)$ is obtained by a successive application of the operators K_m starting from $J_0(x; q, t) = 1$:

$$(4) \quad J_\lambda(x; q, t) = (K_n)^{\lambda_n} (K_{n-1})^{\lambda_{n-1} - \lambda_n} \dots (K_1)^{\lambda_1 - \lambda_2} (1).$$

From this expression, we can show that $J_\lambda(x; q, t)$ is a linear combination of monomial symmetric functions with coefficients in $\mathbb{Z}[q, t]$. Furthermore we have

Theorem B. *For any partition λ and μ , the double Kostka coefficient $K_{\lambda, \mu}(q, t)$ is a polynomial in q and t with integral coefficients.*

(See Theorem 2.4.) Theorem B gives a partial affirmative answer to the conjecture of Macdonald proposed in [Ma], (VI.8.18?) (apart from the positivity of the coefficients).

After some preliminaries on Macdonald's q -difference operators D_r , we formulate our main results in Section 2 and show how Theorem A implies Theorem B. We will propose in Section 3 some determinantal formulas related to our raising operators K_m^\pm . The proof of Theorem A will be given in Section 4 by analyzing the action of the operators K_m^\pm on the generating function of Macdonald polynomials. In Section 5, we will include a similar construction of lowering operators for Macdonald polynomials.

Notes: In [KN], we constructed the raising operators for Macdonald polynomials by means of the Dunkl operators due to I. Cherednik, in an analogous way as L. Lapointe and L. Vinet [LV1, LV2] did for Jack polynomials. We also gave an application to the integrality of transition coefficients of Macdonald polynomials. Although these raising operators involve reflection operators, they act as q -difference operators on symmetric functions; the explicit forms of the corresponding q -difference operators are also determined in [KN]. After this work, we found a direct, elementary proof of the fact that the q -difference operators in question have the property of raising operators for Macdonald polynomials. In this paper, we begin with introducing the raising operators

directly, and show in an elementary way (without affine Hecke algebras and Dunkl operators) that they are the raising operators for Macdonald polynomials that we want. In view of its elementary nature, we decided to make the present paper as self-contained as possible, and also to repeat here the proof of integrality of transition coefficients for the sake of reference. For this reason, this paper has some intersection with our previous paper [KN] (Section 1 and some part of Section 2). From the viewpoint of this paper, our previous paper could be thought of as explaining the meaning of the q -difference raising operators in relation to affine Hecke algebras and Dunkl operators.

§1: Macdonald's q -difference operators.

In this section, we will make a brief review of some basic properties of the Macdonald polynomials (associated with the root system of type A_{n-1} , or the symmetric functions with two parameters) and the commuting family of q -difference operators which have Macdonald polynomials as joint eigenfunctions. For details, see Macdonald's book [Ma].

Let $\mathbb{K} = \mathbb{Q}(q, t)$ be the field of rational functions in two indeterminates q, t and consider the ring $\mathbb{K}[x] = \mathbb{K}[x_1, \dots, x_n]$ of polynomials in n variables $x = (x_1, \dots, x_n)$ with coefficients in \mathbb{K} . Under the natural action of the symmetric group $W = \mathfrak{S}_n$ of degree n , the subring of all symmetric polynomials will be denoted by $\mathbb{K}[x]^W$.

The *Macdonald polynomials* $P_\lambda(x) = P_\lambda(x; q, t)$ (associated with the root system of type A_{n-1}) are symmetric polynomials parametrized by the *partitions* $\lambda = (\lambda_1, \dots, \lambda_n)$ ($\lambda_i \in \mathbb{Z}$, $\lambda_1 \geq \dots \geq \lambda_n \geq 0$). They form a \mathbb{K} -basis of the invariant ring $\mathbb{K}[x]^W$ and are characterized as the joint eigenfunctions of a commuting family of q -difference operators $\{D_r\}_{r=0}^n$. For each $r = 0, 1, \dots, n$, the q -difference operator D_r is defined by

$$(1.1) \quad D_r = t^{\binom{r}{2}} \sum_{\substack{I \subset [1, n] \\ |I|=r}} \prod_{\substack{i \in I \\ j \notin I}} \frac{tx_i - x_j}{x_i - x_j} \prod_{i \in I} T_{q, x_i},$$

where T_{q, x_i} stands for the q -shift operator in the variable x_i : $(T_{q, x_i} f)(x_1, \dots, x_n) = f(x_1, \dots, qx_i, \dots, x_n)$. The summation in (1.1) is taken over all subsets I of the interval $[1, n] = \{1, 2, \dots, n\}$ consisting of r elements. Note that $D_0 = 1$ and $D_n = t^{\binom{n}{2}} T_{q, x_1} \cdots T_{q, x_n}$. Introducing a parameter u , we will use the generating function

$$(1.2) \quad D_x(u) = \sum_{r=0}^n (-u)^r D_r$$

of these operators $\{D_r\}_{r=0}^n$. Note that the operator $D_x(u)$ has the determinantal expression

$$(1.3) \quad D_x(u) = \frac{1}{\Delta(x)} \det(x_j^{n-i} (1 - ut^{n-i} T_{q, x_j}))_{1 \leq i, j \leq n},$$

where $\Delta(x) = \prod_{1 \leq i < j \leq n} (x_i - x_j)$ is the difference product of x_1, \dots, x_n . It is well known that the q -difference operators D_r ($0 \leq r \leq n$) commute with each other, or equivalently, $[D_x(u), D_x(v)] = 0$. Furthermore the Macdonald polynomial $P_\lambda(x)$ satisfies the q -difference equation

$$(1.4) \quad D_x(u) P_\lambda(x) = d_\lambda^n(u) P_\lambda(x), \quad \text{with} \quad d_\lambda^n(u) = \prod_{i=1}^n (1 - ut^{n-i} q^{\lambda_i}),$$

for each partition $\lambda = (\lambda_1, \dots, \lambda_n)$. Recall that each $P_\lambda(x)$ can be written in the form

$$(1.5) \quad P_\lambda(x) = m_\lambda(x) + \sum u_{\lambda\mu} m_\mu(x) \quad (u_{\lambda\mu} \in \mathbb{K}),$$

where, for each partition μ , $m_\mu(x)$ stands for the monomial symmetric function of type μ , and \leq is the dominance order of partitions. The Macdonald polynomials $P_\lambda(x)$ are determined uniquely by the conditions (1.4) and (1.5). The “integral form” $J_\lambda(x) = J_\lambda(x; q, t)$ is defined by the normalization

$$(1.6) \quad J_\lambda(x) = c_\lambda P_\lambda(x), \quad c_\lambda = \prod_{s \in \lambda} (1 - t^{\ell(s)+1} q^{a(s)}),$$

where for each square $s = (i, j)$ in the diagram of a partition λ , the numbers $\ell(s) = \lambda'_j - i$ and $a(s) = \lambda_i - j$ are respectively the *leg-length* and the *arm-length* of λ at $s \in \lambda$ ([Ma], (VI.6.19)).

We recall now the “reproducing kernel” for the Macdonald polynomials. Consider another set of variables $y = (y_1, \dots, y_m)$ and assume that $m \leq n$. We define the function $\Pi(x, y) = \Pi(x, y; q, t)$ by

$$(1.7) \quad \Pi(x, y) = \prod_{\substack{i \in [1, n] \\ j \in [1, m]}} \frac{(tx_i y_j; q)_\infty}{(x_i y_j; q)_\infty},$$

where $(x; q)_\infty = \prod_{k=0}^{\infty} (1 - xq^k)$. The convergence of the infinite product above may be understood in the sense of formal power series. It is known that the function $\Pi(x, y)$ has the expression

$$(1.8) \quad \Pi(x, y) = \sum_{\ell(\lambda) \leq m} b_\lambda P_\lambda(x) P_\lambda(y) \quad (b_\lambda \in \mathbb{K}),$$

where the summation is taken over all partitions λ with length $\leq m$, and each partition $\lambda = (\lambda_1, \dots, \lambda_m, 0, \dots, 0)$ with $\ell(\lambda) \leq m$ is identified with the truncation $(\lambda_1, \dots, \lambda_m)$ when it is used as the suffix for $P_\lambda(y)$. The coefficients b_λ in (1.8) are determined as

$$(1.9) \quad b_\lambda = \prod_{s \in \lambda} \frac{1 - t^{\ell(s)+1} q^{a(s)}}{1 - t^{\ell(s)} q^{a(s)+1}}.$$

We remark that, by (1.4), expression (1.8) is equivalent to the formula

$$(1.10) \quad D_x(u) \Pi(x, y) = (u; t)_{n-m} D_y(ut^{n-m}) \Pi(x, y).$$

Since

$$(1.11) \quad D_x(u) = \sum_{I \subset [1, n]} (-u)^{|I|} t^{\binom{|I|}{2}} \prod_{\substack{i \in I \\ j \notin I}} \frac{tx_i - x_j}{x_i - x_j} \prod_{i \in I} T_{q, x_i},$$

We have

$$(1.12) \quad D_x(u) \Pi(x, y) = \Pi(x, y) F(u; x, y),$$

where

$$(1.13) \quad F(u; x, y) = \sum_{I \subset [1, n]} (-u)^{|I|} t^{\binom{|I|}{2}} \prod_{\substack{i \in I \\ j \notin I}} \frac{tx_i - x_j}{x_i - x_j} \prod_{\substack{i \in I \\ k \in [1, m]}} \frac{1 - x_i y_k}{1 - tx_i y_k}.$$

Hence formula (1.10) is also equivalent to

$$(1.14) \quad F(u; x, y) = (u; t)_{n-m} F(ut^{n-m}; y, x).$$

(See also [MN].)

§2: q -Difference raising operators and transition coefficients.

We now define the q -difference operators K_m^+ and K_m^- ($m = 0, 1, \dots, n$) as follows:

(2.1)

$$K_m^+ = \sum_{\substack{J \subset [1, n] \\ |J|=m}} \prod_{j \in J} x_j \sum_{I \subset J} (-t^{m-n+1})^{|I|} t^{\binom{|I|}{2}} \prod_{\substack{i \in I \\ j \in [1, n] \setminus I}} \frac{tx_i - x_j}{x_i - x_j} \prod_{i \in I} T_{q, x_i},$$

$$K_m^- = \sum_{\substack{J \subset [1, n] \\ |J|=m}} \prod_{j \in J} x_j \sum_{I \subset J} (-t)^{m-|I|} t^{\binom{m-|I|}{2}} \prod_{\substack{i \in I \\ j \in [1, n] \setminus I}} \frac{x_i - tx_j}{x_i - x_j} \prod_{j \in [1, n] \setminus I} T_{q, x_j}.$$

These operators are *dual* to each other in the sense that

$$(2.2) \quad K_m^- = (-t)^m t^{\binom{m}{2}} (K_m^+)^{\vee} T_{q, x_1} \cdots T_{q, x_n}.$$

where the superscript $(\cdot)^{\vee}$ stands for the involution on q -difference operators induced from the transformation $q \rightarrow q^{-1}$, $t \rightarrow t^{-1}$ (cf. [Ma], (VI.8.5)). If $m = 0$ or $m = n$, the operators K_m^{\pm} reduce to

(2.3)

$$K_0^+ = 1, \quad K_n^+ = x_1 \cdots x_n D_x(t) \quad \text{and} \quad K_0^- = T_{q, x_1} \cdots T_{q, x_n}, \quad K_n^- = x_1 \cdots x_n D_x(t).$$

We remark first that these q -difference operators are W -invariant since their definition do not depend on the ordering of the coordinates x_1, \dots, x_n . Hence they transform symmetric functions to symmetric functions. Note also that the coefficients of the operator $\Delta(x)K_m^+$ are in $\mathbb{Z}[t^{\pm 1}][x]$ and that those of $\Delta(x)K_m^-$ are in $\mathbb{Z}[t][x]$. If R is any subring of $\mathbb{K} = \mathbb{Q}(q, t)$ and $f(x)$ is an alternating polynomial in $R[x]$, then the symmetric polynomial $\Delta(x)^{-1}f(x)$ also have coefficients in R , i.e. $\Delta(x)^{-1}f(x) \in R[x]^W$. From these remarks, we obtain

Proposition 2.1. *The q -difference operators K_m^{\pm} ($m = 0, 1, \dots, n$) preserve the ring of symmetric polynomials $\mathbb{K}[x]^W$. Furthermore, the operators K_m^- preserve the ring $\mathbb{Z}[q, t][x]^W$ of symmetric polynomials with coefficients in $\mathbb{Z}[q, t]$.*

We now state our main theorem.

Theorem 2.2. *For any partition λ with $\ell(\lambda) \leq m$ ($m = 0, \dots, n$), we have*

$$(2.4) \quad K_m^+ J_{\lambda}(x) = J_{\lambda+(1^m)}(x) \quad \text{and} \quad K_m^- J_{\lambda}(x) = J_{\lambda+(1^m)}(x).$$

Note that $\lambda + (1^m) = (\lambda_1 + 1, \dots, \lambda_m + 1, 0, \dots, 0)$ if $\ell(\lambda) \leq m$. When $\lambda = 0$, we have $J_{(1^m)}(x) = e_m(x)(t; t)_m$, $e_m(x)$ being the elementary symmetric function of degree m . Hence, formulas in (2.4) imply the following:

$$(2.5) \quad \sum_{\substack{J \subset [1, n] \\ |J|=m}} \prod_{j \in J} x_j \sum_{I \subset J} (-t^{m-n+1})^{|I|} t^{\binom{|I|}{2}} \prod_{\substack{i \in I \\ j \notin I}} \frac{tx_i - x_j}{x_i - x_j} = e_m(x)(t; t)_m,$$

$$\sum_{\substack{J \subset [1, n] \\ |J|=m}} \prod_{j \in J} x_j \sum_{I \subset J} (-t)^{m-|I|} t^{\binom{m-|I|}{2}} \prod_{\substack{i \in I \\ j \notin I}} \frac{x_i - tx_j}{x_i - x_j} = e_m(x)(t; t)_m.$$

Theorem 2.2 will be proved later in Section 4. In this section, we will discuss some consequences of our theorem.

From Theorem 2.2, it follows that the Macdonald polynomial $J_\lambda(x)$ for any partition λ can be obtained from the constant function $J_0(x) = 1$ by a successive application of the raising operators $K_m = K_m^+$ or K_m^- . Namely we have

$$(2.6) \quad J_\lambda(x) = (K_n)^{\lambda_n} (K_{n-1})^{\lambda_{n-1}-\lambda_n} \dots (K_1)^{\lambda_1-\lambda_2}(1),$$

or equivalently

$$(2.7) \quad J_\lambda(x) = K_{\mu_1} K_{\mu_2} \dots K_{\mu_s}(1),$$

where $\mu = (\mu_1, \dots, \mu_s)$ is the conjugate partition λ' of λ . Since the operators K_m^- preserve the ring $\mathbb{Z}[q, t][x]^W$, we have $J_\lambda(x) \in \mathbb{Z}[q, t]^W$. In other words,

Theorem 2.3. *For any partition λ , the Macdonald polynomial $J_\lambda(x)$ is expressed as a linear combination of monomial symmetric functions $m_\mu(x)$ with coefficients in $\mathbb{Z}[q, t]$.*

Note that this statement is valid for any n . Hence it implies that the transition coefficients between $J_\lambda(x)$ and the monomial symmetric functions *in infinite variables* are polynomials in q and t with integral coefficients.

From this theorem, it is not difficult to conclude the integrality of the so-called double Kostka coefficients. Recall that the *double Kostka coefficients* (or (q, t) -Kostka coefficients) $K_{\lambda, \mu}(q, t)$ are defined as the transition coefficients between the integral forms of Macdonald polynomials and the *big Schur functions*:

$$(2.8) \quad J_\mu(x; q, t) = \sum_{\lambda} K_{\lambda, \mu}(q, t) S_\lambda(x; t),$$

while Theorem 2.3 is concerned with the transition coefficients with the monomial symmetric functions. To be more precise, (2.8) should be understood as an equality in infinite variables. For the definition of $S_\lambda(x; t)$, we refer to [Ma], (III.4.5).

Theorem 2.4. *For any partitions λ and μ , the double Kostka coefficient $K_{\lambda, \mu}(q, t)$ is a polynomial in q and t with integral coefficients.*

Theorem 2.4 gives a partial affirmative answer to the conjecture of Macdonald proposed in [Ma], (VI.8.18?), apart from the positivity of the coefficients.

Proof of Theorem 2.4. As in [Ma], p.241, the transition coefficients between monomial symmetric functions and big Schur functions have the form $p(t)/q(t)$, where $p(t), q(t) \in \mathbb{Z}[t]$ and $q(0) = 1$. Note in particular that they belong to the ring $\mathbb{Q}[t]_{(t)}$ of rational functions in t , regular at $t = 0$. Combining this fact with Theorem 2.3, we see that each double Kostka coefficients can be written as a finite sum of the form

$$(2.9) \quad K_{\lambda, \mu}(q, t) = \sum_{k \geq 0} \frac{p_{\lambda, \mu}^{(k)}(t)}{q_{\lambda, \mu}^{(k)}(t)} q^k,$$

where all $p_{\lambda, \mu}^{(k)}(t)$ and $q_{\lambda, \mu}^{(k)}(t)$ belong to $\mathbb{Z}[t]$ and $q_{\lambda, \mu}^{(k)}(0) = 1$. In particular we have $K_{\lambda, \mu}(q, t) \in \mathbb{Q}[t]_{(t)}[q]$. We can now apply the duality with respect to q and t again, between $I_{\lambda}^{(q, t)}(x; t)$ and $I_{\lambda}^{(q, t)}(x; q)$ and between $S_{\lambda}(x; t)$ and $S_{\lambda}(x; q)$ to

conclude $K_{\lambda,\mu}(q, t) = K_{\lambda',\mu'}(t, q)$ ([Ma], (VI.8.15)). Hence, we have $K_{\lambda,\mu}(q, t) \in \mathbb{Q}[t]_{(t)}[q] \cap \mathbb{Q}[q]_{(q)}[t]$. By using Taylor expansions at $t = q = 0$, one can easily see that the intersection of the two subalgebras $\mathbb{Q}[t]_{(t)}[q]$ and $\mathbb{Q}[q]_{(q)}[t]$ coincides precisely with $\mathbb{Q}[q, t]$. Hence we have $K_{\lambda,\mu}(q, t) \in \mathbb{Q}[q, t]$. In expression (2.9), it means that, for any k , $p_{\lambda\mu}^{(k)}(t)/q_{\lambda\mu}^{(k)}(t) \in \mathbb{Q}[t]$, i.e., $q_{\lambda\mu}^{(k)}(t)$ divides $p_{\lambda\mu}^{(k)}(t)$. Since $q_{\lambda\mu}^{(k)}(0) = 1$, it follows that $p_{\lambda\mu}^{(k)}(t)/q_{\lambda\mu}^{(k)}(t) \in \mathbb{Z}[t]$ for all k . Namely we have $K_{\lambda,\mu}(q, t) \in \mathbb{Z}[q, t]$. \square

Our raising operators have different expressions, from which one can take the *quasi-classical* limits as q tends to 1.

Proposition 2.5. *For each $m = 0, 1, \dots, n$, we have*

$$(2.10) \quad K_m^+ = \frac{1}{\Delta(x)} \sum_{w \in W} \epsilon(w) w \left(x_1^{\delta_1} \cdots x_n^{\delta_n} e_m(X_1, \dots, X_n) \right)$$

where $\delta_i = n - i$ ($i = 1, \dots, n$) and $e_m(X_1, \dots, X_n)$ stands for the m -th elementary symmetric function of the q -difference operators

$$(2.11) \quad X_i = x_i(1 - t^{m-i+1}T_{q,x_i}) \quad (i = 1, \dots, n).$$

Similarly we have

$$(2.12) \quad K_m^- = \frac{t^{\binom{n}{2} - \binom{n-m}{2}}}{\Delta(x)} \sum_{w \in W} \epsilon(w) w \left(x_1^{\delta_1} \cdots x_n^{\delta_n} e_m(X'_1, \dots, X'_n) \right) T_{q,x_1} \cdots T_{q,x_n}$$

where $X'_i = X_i t^{-\delta_i} T_{q,x_i}^{-1}$ ($i = 1, \dots, n$).

The formulas of Proposition 2.5 will be proved in Section 3 by using certain *determinantal formulas* for the raising operators K_m^\pm .

Let $q = t^\alpha$ and let $t \rightarrow 1$. Then the limits of the two q -difference operators K_m^+ and K_m^- give rise to a same differential operator:

$$(2.13) \quad \begin{aligned} \mathcal{K}_m &= \lim_{\substack{q=t^\alpha \\ t \rightarrow 1}} \frac{K_m^+}{(1-t)^m} = \lim_{\substack{q=t^\alpha \\ t \rightarrow 1}} \frac{K_m^-}{(1-t)^m} \\ &= \frac{1}{\Delta(x)} \sum_{w \in W} \epsilon(w) w \left(x_1^{\delta_1} \cdots x_n^{\delta_n} e_m(\mathcal{X}_1, \dots, \mathcal{X}_n) \right), \end{aligned}$$

where

$$(2.14) \quad \mathcal{X}_i = x_i \left(\alpha x_i \frac{\partial}{\partial x_i} + (m - i + 1) \right) \quad (i = 1, \dots, n).$$

For each partition λ , let $J_\lambda^{(\alpha)}(x)$ be the Jack polynomial normalized so that the coefficient of $m_\lambda(x)$ becomes $c_\lambda(\alpha) = \prod_{s \in \lambda} (\alpha a(s) + \ell(s) + 1)$, as in [Ma], (VI.10.22). Since

$$(2.15) \quad J_\lambda^{(\alpha)}(x) = \lim_{t \rightarrow 1} \frac{J_\lambda(x; t^\alpha, t)}{(1-t)^{|\lambda|}},$$

Theorem 2.2, combined with (2.12), (2.15), implies the following

Theorem 2.6. *The differential operators \mathcal{K}_m ($m = 0, 1, \dots, n$) in (2.13) are raising operators for the Jack polynomials $J_\lambda^{(\alpha)}(x)$ in the sense that*

$$(2.16) \quad \mathcal{K}_m J_\lambda^{(\alpha)}(x) = J_{\lambda+(1^m)}^{(\alpha)}(x)$$

for any partition λ with $\ell(\lambda) \leq m$.

By a similar argument as we proved Theorem 2.3, we obtain

Corollary. *For each partition λ , the Jack polynomial $J_\lambda^{(\alpha)}(x)$ is expressed as a linear combination of monomial symmetric functions with coefficients in $\mathbb{Z}[\alpha]$.*

Theorem 2.6 can be regarded as a differential version (without reflection operators) of the raising operators of Lapointe-Vinet [LV1, LV2].

§3: Determinantal formulas.

In this section we propose determinantal formulas for some q -difference operators related to our raising operators. The proof of Proposition 2.5 is an application of these determinant formulas. Some part of this section will be used also in the proof of Theorem 2.2 in Section 4.

Recall that Macdonald's q -difference operator $D_x(u)$ has the determinantal formula

$$(3.1) \quad D_x(u) = \frac{1}{\Delta(x)} \det(x_j^{n-i} (1 - ut^{n-i} T_{q, x_j}))_{1 \leq i, j \leq n}.$$

We begin with a comment on this noncommutative determinant. Suppose an $n \times n$ matrix $A = (a_{ij})_{i,j}$ with entries in a noncommutative algebra has the property that $a_{ij}a_{k\ell} = a_{k\ell}a_{ij}$ for $i \neq k$ and $j \neq \ell$. If this is the case, one has

$$(3.2) \quad a_{\sigma(1)1} \cdots a_{\sigma(n)n} = a_{1\tau(1)} \cdots a_{n\tau(n)} \quad (\tau = \sigma^{-1})$$

for any permutation σ , since the n elements $a_{\sigma(1)1}, \dots, a_{\sigma(n)n}$ commute with each other. Hence we do not need to distinguish the *column* determinant and the *row* determinant:

$$(3.3) \quad \det(A) = \sum_{\sigma \in \mathfrak{S}_n} a_{\sigma(1)1} \cdots a_{\sigma(n)n} = \sum_{\sigma \in \mathfrak{S}_n} a_{1\sigma(1)} \cdots a_{n\sigma(n)}.$$

Since the matrix of (3.1) has this property, and there is no ambiguity in the notation. We generalize this type of determinantal formula from the viewpoint of raising operators.

In view of the expression (2.1) of our raising operators, let us consider the following q -difference operators including a parameter u :

$$(3.4) \quad \begin{aligned} K_m(u) &= \sum_{\substack{J \subset [1, n] \\ |J|=m}} x_J \sum_{I \subset J} (-u)^{|I|} \frac{T_{t,x}^I(\Delta(x))}{\Delta(x)} T_{q,x}^I, \\ L_m(u) &= \sum_{J \subset [1, n]} x_J \sum_{I \subset J} (-u)^{m-|I|} \frac{T_{t,x}^{I^c}(\Delta(x))}{\Delta(x)} T_{q,x}^{I^c}, \end{aligned}$$

for each $m = 0, 1, \dots, n$, where we have used the abbreviation

$$(3.5) \quad x_J = \prod_{j \in J}, \quad T_{q,x}^I = \prod_{i \in I} T_{q,x_i}, \quad T_{q,x}^{I^c} = \prod_{j \in [1,n] \setminus I} T_{q,x_j}.$$

Since

$$(3.6) \quad \frac{T_{t,x}^I(\Delta(x))}{\Delta(x)} = t^{\binom{|I|}{2}} \prod_{\substack{i \in I \\ j \in [1,n] \setminus I}} \frac{tx_i - x_j}{x_i - x_j},$$

the q -difference operators K_m^\pm are recovered as the special cases

$$(3.7) \quad K_m^+ = K_m(t^{m-n+1}), \quad K_m^- = t^{-\binom{n-m}{2}} L_m(t^{m-n+1}).$$

Introducing another parameter v , we set

$$(3.8) \quad K(u, v) = \sum_{m=0}^n v^m K_m(u), \quad L(u, v) = \sum_{m=0}^n v^m L_m(u).$$

Then we have the following determinantal formula for $K(u, v)$ and $L(u, v)$.

Proposition 3.1.

$$(3.9) \quad \begin{aligned} K(u, v) &= \frac{1}{\Delta(x)} \det(x_j^{\delta_i} (1 + vx_j(1 - ut^{\delta_i} T_{q,x_j})))_{1 \leq i, j \leq n}, \\ L(u, v) &= \frac{1}{\Delta(x)} \det(x_j^{\delta_i} (t^{\delta_i} T_{q,x_j} + vx_j(1 - ut^{\delta_i} T_{q,x_j})))_{1 \leq i, j \leq n}, \end{aligned}$$

where $\delta_i = n - i$ for $i = 1, \dots, n$. When $q = t$, these formulas reduce to

$$(3.10) \quad \begin{aligned} K(u, v) &= \frac{1}{\Delta(x)} \prod_{j=1}^n (1 + vx_j(1 - uT_{t,x_j})) \Delta(x), \\ L(u, v) &= \frac{1}{\Delta(x)} \prod_{j=1}^n (T_{t,x_j} + vx_j(1 - uT_{t,x_j})) \Delta(x). \end{aligned}$$

If we take the coefficients of v^n , formula (3.9) recovers formula (3.1) for $D_x(u)$. The leading coefficient in v of formula (3.10) also gives the well-known formula $D_x(u) = \Delta(x)^{-1} \prod_{i=1}^n (1 - uT_{t,x_i}) \Delta(x)$ when $q = t$.

Before the proof of Proposition 3.1, we will give a remark on the symbol of a q -difference operator. For a given q -difference operator

$$(3.11) \quad P(x; T_{q,x}) = \sum_{\mu} a_{\mu}(x) T_{q,x}^{\mu} \in \mathbb{K}(x)[T_{q,x}^{\pm 1}]$$

with rational coefficients, we define the *symbol* $p(x; \xi)$ of $P(x; T_{q,x})$ to be the function

$$(3.12) \quad p(x; \xi) = \sum a_{\mu}(x) \xi^{\mu} \in \mathbb{K}(x)[\xi^{\pm}]$$

in the commuting variables $(x; \xi)$. Then the action of $P(x; T_{q,x})$ on a function $\varphi(x)$ is recovered by

$$(3.13) \quad P(x; T_{q,x})\varphi(x) = p(x; T_{q,y})\varphi(y)|_{y=x},$$

by using the duplicated variables.

Proof of Proposition 3.1. We will treat only the case of $K(u, v)$, since the same argument gives the corresponding results for $L(u, v)$. Note first that the symbol of the operator $K_m(u)$ is given by

$$(3.14) \quad \begin{aligned} \text{Symb}(K_m(u)) &= \frac{1}{\Delta(x)} \sum_{|J|=m} x_J \sum_{I \subset J} (-u)^{|I|} T_{t,x}^I(\Delta(x)) \xi_I \\ &= \frac{1}{\Delta(x)} \sum_{|J|=m} x_J \prod_{j \in J} (1 - uT_{t,x_j} \xi_j) \Delta(x). \end{aligned}$$

Hence we have

$$(3.15) \quad \begin{aligned} \text{Symb}(K(u, v)) &= \frac{1}{\Delta(x)} \sum_J v^{|J|} x_J \prod_{j \in J} (1 - uT_{t,x_j} \xi_j) \Delta(x) \\ &= \frac{1}{\Delta(x)} \prod_{j=1}^n (1 + vx_j(1 - uT_{t,x_j} \xi_j))(\Delta(x)). \end{aligned}$$

When $q = t$, this already implies

$$(3.16) \quad K(u, v)\varphi(x) = \frac{1}{\Delta(x)} \prod_{j=1}^n (1 + vx_j(1 - uT_{t,x_j}))(\Delta(x)\varphi(x)),$$

since the operators $x_j(1 - uT_{t,x_j} \xi_j)$ ($j = 1, \dots, n$) commute with each other. This proves formula (3.10) for $K(u, v)$. Returning to the general setting, we rewrite formula (3.15) as follows:

$$(3.17) \quad \begin{aligned} \text{Symb}(K(u, v)) &= \frac{1}{\Delta(x)} \sum_{\sigma \in \mathfrak{S}_n} \epsilon(\sigma) \prod_{j=1}^n (1 + vx_j(1 - uT_{t,x_j} \xi_j))(x_1^{\delta_{\sigma(1)}} \cdots x_n^{\delta_{\sigma(n)}}) \\ &= \frac{1}{\Delta(x)} \sum_{\sigma \in \mathfrak{S}_n} \epsilon(\sigma) \prod_{j=1}^n x_j^{\delta_{\sigma(j)}} (1 + vx_j(1 - ut^{\delta_{\sigma(j)}} \xi_j)) \\ &= \frac{1}{\Delta(x)} \det(x_j^{\delta_i} (1 + vx_j(1 - ut^{\delta_i} \xi_j)))_{1 \leq i, j \leq n}. \end{aligned}$$

Consider now the determinant of Proposition:

$$(3.18) \quad \begin{aligned} &\frac{1}{\Delta(x)} \det(x_j^{\delta_i} (1 + vx_j(1 - ut^{\delta_i} T_{q,x_j})))_{1 \leq i, j \leq n} \\ &= \frac{1}{\Delta(x)} \sum_{\sigma \in \mathfrak{S}_n} \epsilon(\sigma) \prod_{j=1}^n x_j^{\delta_{\sigma(j)}} (1 + vx_j(1 - ut^{\delta_{\sigma(j)}} T_{q,x_j})) \end{aligned}$$

Note that the factors in the product of the last expression are already ordered normally and that they act on different variables. From this we see that the symbol of the operator of (3.18) is precisely given by the formula (3.17). This implies formula (3.9) of Proposition. \square

Proposition 2.5 is obtained by expanding the determinantal formulas for $K(u, v)$ and $L(u, v)$ with respect to v . In fact we have

$$(3.19) \quad \begin{aligned} K(u, v) &= \sum_{w \in W} \epsilon(w) w \left(x_1^{\delta_1} \cdots x_n^{\delta_n} \prod_{j=1}^n (1 + vx_j(1 - ut^{\delta_j} T_{q, x_j})) \right) \\ &= \sum_{m=0}^n v^m \sum_{w \in W} \epsilon(w) w \left(x_1^{\delta_1} \cdots x_n^{\delta_n} e_m(X_1(u), \dots, X_n(u)) \right), \end{aligned}$$

where we set

$$(3.20) \quad X_j(u) = x_j(1 - ut^{\delta_j} T_{q, x_j}) \quad (j = 1, \dots, n).$$

Namely we have

$$(3.21) \quad K_m(u) = \sum_{w \in W} \epsilon(w) w \left(x_1^{\delta_1} \cdots x_n^{\delta_n} e_m(X_1(u), \dots, X_n(u)) \right).$$

Formula (2.10) for K_m^+ is the special case when $u = t^{m-n+1}$. A similar computation for $L(u, v)$ gives formula (2.10) for $K_m^- = t^{-\binom{n-m}{2}} L_m(t^{m-n+1})$.

§4: Proof of Theorem 2.2.

The main idea in our proof of Theorem 2.2 is to use the generating function

$$(4.1) \quad \Pi(x, y) = \prod_{\substack{i \in [1, n] \\ j \in [1, m]}} \frac{(tx_i y_j; q)_\infty}{(x_i y_j; q)_\infty} = \sum_{\ell(\lambda) \leq m} b_\lambda P_\lambda(x) P_\lambda(y)$$

for the Macdonald polynomials in $x = (x_1, \dots, x_n)$ and in $y = (y_1, \dots, y_m)$, where $0 \leq m \leq n$.

Lemma 4.1. *Let $K_m : \mathbb{K}[x]^W \rightarrow \mathbb{K}[x]^W$ be an operator acting on symmetric functions. Then K_m has the property that*

$$(4.2) \quad K_m J_\lambda(x) = J_{\lambda+(1^m)}(x) \quad \text{for any partition } \lambda \text{ with } \ell(\lambda) \leq m,$$

if and only if the following equality is verified:

$$(4.3) \quad K_{m,x} \Pi(x, y) = \frac{1}{D_y(1)} D_y(1) \Pi(x, y),$$

where the suffix x in $K_{m,x}$ indicates that K_m should act on the x variables.

Proof. In fact, we have

$$(4.4) \quad \begin{aligned} y_1 \cdots y_m K_{m,x} \Pi(x, y) &= \sum_{\ell(\lambda) \leq m} b_\lambda K_m(P_\lambda(x)) y_1 \cdots y_m P_\lambda(y) \\ &= \sum_{\ell(\lambda) \leq m} b_\lambda K_m(P_\lambda(x)) P_{\lambda+(1^m)}(y). \end{aligned}$$

As to the action of $D_y(1)$, we have

$$(4.5) \quad \begin{aligned} D_y(1) \Pi(x, y) &= \sum_{\ell(\lambda) \leq m} b_\lambda P_\lambda(x) d_\lambda^m(1) P_\lambda(y) \\ &= \sum_{\ell(\lambda) \leq m} d_{\lambda+(1^m)}^m(1) b_{\lambda+(1^m)} P_{\lambda+(1^m)}(x) P_{\lambda+(1^m)}(y), \end{aligned}$$

since $d_\lambda^m(1) = (1-t^{m-1}q^{\lambda_1}) \cdots (1-tq^{\lambda_{m-1}})(1-q^{\lambda_m}) = 0$ if $\lambda_m = 0$. Hence, equality (4.3) is equivalent to

$$(4.6) \quad b_\lambda K_m P_\lambda(x) = d_{\lambda+(1^m)}^m(1) b_{\lambda+(1^m)} P_{\lambda+(1^m)}(x) \quad \text{for any } \lambda \text{ with } \ell(\lambda) \leq m.$$

This is equivalent to property (4.2) since $J_\lambda(x) = c_\lambda P_\lambda(x)$ and

$$(4.7) \quad \frac{b_\lambda}{c_\lambda} = \frac{b_{\lambda+(1^m)}}{c_{\lambda+(1^m)}} d_{\lambda+(1^m)}^m(1)$$

by (1.4), (1.6) and (1.9). \square

We now prove the equality (4.3) for our operators $K_m = K_m^\pm$ defined in (2.1). We will mainly consider the case of $K_m = K_m^+$:

$$(4.8) \quad K_{m,x}^+ \Pi(x, y) = \frac{1}{y_1 \cdots y_m} D_y(1) \Pi(x, y),$$

since the same argument is valid for K_m^- as well. Firstly, the two sides of (4.8) can be written as follows:

$$(4.9) \quad \begin{aligned} &K_{m,x}^+ \Pi(x, y) \\ &= \Pi(x, y) \sum_{\substack{J \subset [1,n] \\ |J|=m}} x_J \sum_{I \subset J} (-t^{m-n+1})^{|I|} \frac{T_{t,x}^I(\Delta(x))}{\Delta(x)} \prod_{\substack{i \in I \\ k \in [1,m]}} \frac{1 - x_i y_k}{1 - t x_i y_k} \end{aligned}$$

and

$$(4.10) \quad \begin{aligned} &\frac{1}{y_1 \cdots y_m} D_y(1) \Pi(x, y) \\ &= \Pi(x, y) \frac{1}{y_1 \cdots y_m} \sum_{K \subset [1,m]} (-1)^{|K|} \frac{T_{t,y}^K(\Delta(y))}{\Delta(y)} \prod_{i \in [1,n]} \frac{1 - x_i y_k}{1 - t x_i y_k}. \end{aligned}$$

Hence equality (4.8) is equivalent to

$$(4.11) \quad \sum_{\substack{J \subset [1, n] \\ |J|=m}} x_J \sum_{I \subset J} (-t^{m-n+1})^{|I|} t^{\binom{|I|}{2}} \prod_{\substack{i \in I \\ j \notin I}} \frac{tx_i - x_j}{x_i - x_j} \prod_{\substack{i \in I \\ k \in [1, m]}} \frac{1 - x_i y_k}{1 - tx_i y_k} \\ = \frac{1}{y_1 \cdots y_m} \sum_{K \subset [1, m]} (-1)^{|K|} t^{\binom{|K|}{2}} \prod_{\substack{k \in K \\ \ell \notin K}} \frac{ty_k - y_\ell}{y_k - y_\ell} \prod_{\substack{i \in [1, n] \\ k \in K}} \frac{1 - x_i y_k}{1 - tx_i y_k}.$$

The key point now is that this equality does *not* depend on q . It means that, in order to prove (4.8), we have only to establish it for some special value of q .

From now on, we consider the case of $q = t$, so that the Macdonald polynomials $P_\lambda(x; q, t)$ reduce to the Schur functions

$$(4.12) \quad s_\lambda(x) = \frac{\det(x_j^{\lambda_i + \delta_i})_{1 \leq i, j \leq n}}{\Delta(x)}, \quad \lambda = (\lambda_1, \dots, \lambda_n).$$

Setting $q = t$, we have to prove that $K_m^+ J_\lambda(x; t, t) = J_{\lambda+(1^m)}(x; t, t)$, i.e.,

$$(4.13) \quad K_m^+ s_\lambda(x) = s_{\lambda+(1^m)} \prod_{k=1}^m (1 - t^{\lambda_k + m - k + 1})$$

for any partition λ with $\ell(\lambda) \leq m$. It would then imply (4.8) for the case $q = t$ by Lemma 4.1, hence (4.8) for the general (q, t) . With the notation of Section 3, we will compute the action of $K(u, v)$ on Schur functions by means of the formula (3.10):

$$(4.14) \quad K(u, v) = \frac{1}{\Delta(x)} \prod_{j=1}^n (1 + vx_j(1 - uT_{q, x_j})) \Delta(x).$$

In fact, we have

$$(4.15) \quad \begin{aligned} K(u, v) s_\lambda(x) &= \frac{1}{\Delta(x)} \prod_{k=1}^n (1 + vx_k(1 - uT_{t, x_k})) \det(x_j^{\lambda_i + \delta_i}) \\ &= \frac{1}{\Delta(x)} \det(x_j^{\lambda_i + \delta_i} (1 + vx_j(1 - ut^{\lambda_i + \delta_i}))) \\ &= \frac{1}{\Delta(x)} \sum_{K \subset [1, n]} v^{|K|} \det(x_j^{\lambda_i + \delta_i + \theta_K(i)}) \prod_{k \in K} (1 - ut^{\lambda_k + \delta_k}), \end{aligned}$$

where $\theta_K(i) = 1$ if $i \in K$ and $\theta_K(i) = 0$ otherwise. Extending the notation (4.11) to any n -tuple of integers, we can rewrite (4.15) as

$$(4.16) \quad K(u, v) s_\lambda(x) = \sum v^{|K|} s_{\lambda+(1^K)}(x) \prod (1 - ut^{\lambda_k + \delta_k}),$$

where $(1^K) = (\theta_K(i))_{1 \leq i \leq n}$. Suppose now that λ is a partition with $\ell(\lambda) \leq m$. Since

$$(4.17) \quad \lambda + \delta = (\lambda_1 + n - 1, \dots, \lambda_m + n - m, n - m - 1, \dots, 0),$$

it is easily seen that $s_{\lambda+(1^K)}(x) = 0$ unless $K \subset [1, m]$ or $K \cap [m+1, n] = [m+1, m+r]$ for some $1 \leq r \leq n-m$. We specialize the value of u to be $t^{-(n-m-1)}$ so that $\prod_{k \in K} (1 - ut^{\lambda_k + \delta_k}) = 0$ if $K \cap [m+1, n] \neq \emptyset$. Then we see

$$(4.18) \quad K(t^{m-n+1}, v)s_\lambda(x) = \sum_{K \subset [1, m]} s_{\lambda+(1^K)}(x) \prod_{k \in K} (1 - t^{\lambda_k + m - k + 1})$$

Namely we have

$$(4.19) \quad K_\ell(t^{m-n+1})s_\lambda(x) = \sum_{\substack{K \subset [1, m] \\ |K| = \ell}} s_{\lambda+(1^K)}(x) \prod_{k \in K} (1 - t^{\lambda_k + m - k + 1})$$

for any partition λ with $\ell(\lambda) \leq m$. In particular we obtain

$$(4.20) \quad K_m(t^{m-n+1})s_\lambda(x) = s_{\lambda+(1^m)} \prod_{k=1}^m (1 - t^{\lambda_k + m - k + 1}),$$

$$K_\ell(t^{m-n+1})s_\lambda(x) = 0 \quad \text{if } \ell > m.$$

This shows that, when $q = t$, $K_m^+ = K_m(t^{m-n+1})$ has property (4.2). It implies (4.8) for the case $q = t$ by Lemma 4.1, hence (4.8) for the general (q, t) via equality (4.11). Again by Lemma 4.1, we see that, for any (q, t) , the operator K_m^+ has the property (4.2) of raising operators for the Macdonald polynomials $J_\lambda(x)$. This completes the proof of Theorem 2.2 for $K_m = K_m^+$.

The same argument is valid for $K_m = K_m^-$ as well. For comparison, we will include some formulas for the q -difference operators $K_m^- = t^{-\binom{n-m}{2}} L_m(t^{m-n+1})$ and $L(u, v)$. Formula (4.3) for K_m^- is equivalent to

$$(4.21) \quad \sum_{\substack{J \subset [1, n] \\ |J| = m}} x_J \sum_{I \subset J} (-t)^{m-|I|} t^{\binom{m-|I|}{2}} \prod_{\substack{i \in I \\ j \notin I}} \frac{x_i - tx_j}{x_i - x_j} \prod_{\substack{j \notin I \\ k \in [1, m]}} \frac{1 - x_j y_k}{1 - tx_j y_k}$$

$$= \frac{1}{y_1 \cdots y_m} \sum_{K \subset [1, m]} (-1)^{|K|} t^{\binom{|K|}{2}} \prod_{\substack{k \in K \\ \ell \notin K}} \frac{ty_k - y_\ell}{y_k - y_\ell} \prod_{\substack{i \in [1, n] \\ k \in K}} \frac{1 - x_i y_k}{1 - tx_i y_k}.$$

It can be proved by computing the action of $L(u, v)$ with $q = t$ on Schur functions:

$$(4.22) \quad L(u, v)s_\lambda(x) = \sum_{K \subset [1, n]} s_{\lambda+(1^K)}(x) \prod_{k \in K} (1 - ut^{\lambda_k + \delta_k}) \prod_{\ell \notin K} t^{\lambda_\ell + \delta_\ell}.$$

As byproducts of this proof we obtain several interesting formulas. Consider the case $q = t$, and specialize (4.16) and (4.22) to $\lambda = 0$. Then one can easily see that these formulas reduce to

$$(4.23) \quad K(u, v)1 = \sum_{m=0}^n v^m e_m(x)(ut^{n-m}; t)_m,$$

$$L(u, v)1 = \sum_{m=0}^n v^m e_m(x)(ut^{n-m}; t)_m t^{\binom{n-m}{2}}.$$

Proposition 4.2. *For each $m = 0, 1, \dots, m$, one has*

(4.24)

$$\sum_{\substack{J \subset [1, n] \\ |J|=m}} x_J \sum_{I \subset J} (-u)^{|I|} t^{\binom{|I|}{2}} \prod_{\substack{i \in I \\ j \notin I}} \frac{tx_i - x_j}{x_i - x_j} = e_m(x)(ut^{n-m}; t)_m,$$

$$\sum_{\substack{J \subset [1, n] \\ |J|=m}} x_J \sum_{I \subset J} (-u)^{m-|I|} t^{\binom{n-|I|}{2}} \prod_{\substack{i \in I \\ j \notin I}} \frac{x_i - tx_j}{x_i - x_j} = e_m(x)(ut^{n-m}; t)_m t^{\binom{n-m}{2}}.$$

In particular one has

(4.25)

$$\sum_{\substack{J \subset [1, n] \\ |J|=m}} x_J \sum_{\substack{I \subset J \\ |I|=r}} \prod_{\substack{i \in I \\ j \notin I}} \frac{tx_i - x_j}{x_i - x_j} = t^{(n-m)r} \begin{bmatrix} m \\ r \end{bmatrix}_t e_m(x),$$

$$\sum_{\substack{J \subset [1, n] \\ |J|=m}} x_J \sum_{\substack{I \subset J \\ |I|=r}} \prod_{\substack{i \in I \\ j \notin I}} \frac{x_i - tx_j}{x_i - x_j} = \begin{bmatrix} m \\ r \end{bmatrix}_t e_m(x),$$

for each $r = 0, 1, \dots, m$.

Formulas (4.25) are obtained from (4.24) by taking the coefficients of u^r or u^{m-r} .

Our raising operators K_m^\pm for Macdonald polynomials $J_\lambda(x; q, t)$ can be understood as a systematic generalization of formulas of this kind.

§5: q -Difference lowering operators.

By using similar ideas, we can also construct q -difference lowering operators for Macdonald polynomials.

For each $m = 0, 1, \dots, n$, we define the q -difference operators M_m^+ and M_m^- as follows:

(5.1)

$$M_m^+ = \sum_{\substack{J \subset [1, n] \\ |J|=m}} \prod_{j \in J} \frac{1}{x_j} \sum_{I \subset J} (-1)^{|I|} t^{\binom{|I|}{2}} \prod_{\substack{i \in I \\ j \in [1, n] \setminus I}} \frac{tx_i - x_j}{x_i - x_j} \prod_{i \in I} T_{q, x_i},$$

$$M_m^- = \sum_{\substack{J \subset [1, n] \\ |J|=m}} \prod_{j \in J} \frac{1}{x_j} \sum_{I \subset J} (-t^{n-m})^{m-|I|} t^{\binom{m-|I|}{2}} \prod_{\substack{i \in I \\ j \in [1, n] \setminus I}} \frac{x_i - tx_j}{x_i - x_j} \prod_{j \in [1, n] \setminus I} T_{q, x_j}.$$

Theorem 5.1. *For each $m = 0, 1, \dots, n$, the q -difference operators M_m^+ and M_m^- preserve the ring $\mathbb{K}[x]^W$ of symmetric polynomials. Furthermore they are lowering operators for Macdonald polynomials in the sense that*

$$M_m^\pm J_\lambda(x) = a_\lambda^m J_{\lambda - (1^m)}(x), \quad a_\lambda^m = \prod_{i=1}^m (1 - t^{m-i} q^{\lambda_i}) (1 - t^{n-i+1} q^{\lambda_i - 1}),$$

for any partition λ with $\ell(\lambda) \leq m$. In particular one has $M_m^\pm J_\lambda(x) = 0$ if $\ell(\lambda) < m$.

For the proof of Theorem 5.1, we prove the equality

(5.2)
$$M_m^+ \prod_{i=1}^m (1 - t^{m-i} q^{\lambda_i}) = \prod_{i=1}^m (1 - t^{n-i+1} q^{\lambda_i - 1}) M_m^+ J_\lambda(x)$$

for the auxiliary variables $y = (y_1, \dots, y_m)$. By the same argument as in Lemma 4.1, one can prove that equality (5.2) implies Theorem 5.1. Also, equality (5.2) is reduced to the special case when $q = t$.

We now introduce the following q -difference operators $M(u, v)$ and $N(u, v)$:

$$(5.3) \quad M(u, v) = \sum_{m=0}^n v^m M_m(u), \quad N(u, v) = \sum_{m=0}^n v^m N_m(u),$$

where

$$(5.4) \quad M_m(u) = \sum_{\substack{J \subset [1, n] \\ |J|=m}} \frac{1}{x_J} \sum_{I \subset J} (-u)^{|I|} \frac{T_{t,x}^I(\Delta(x))}{\Delta(x)} T_{q,x}^I,$$

$$N_m(u) = \sum_{\substack{J \subset [1, n] \\ |J|=m}} \frac{1}{x_J} \sum_{I \subset J} (-u)^{m-|I|} \frac{T_{t,x}^{I^c}(\Delta(x))}{\Delta(x)} T_{q,x}^{I^c},$$

so that $M_m^+ = M_m(1)$, and $M_m^- = t^{-\binom{n-m}{2}} N_m(1)$. These operators have the determinantal formulas

$$(5.5) \quad M(u, v) = \frac{1}{\Delta(x)} \det(x_j^{\delta_i} (1 + \frac{v}{x_j} (1 - ut^{\delta_i} T_{q,x_j})))_{1 \leq i, j \leq n},$$

$$N(u, v) = \frac{1}{\Delta(x)} \det(x_j^{\delta_i} (t^{\delta_i} T_{q,x_j} + \frac{v}{x_j} (1 - ut^{\delta_i} T_{q,x_j})))_{1 \leq i, j \leq n}.$$

When $q = t$, these formulas reduce to

$$(5.6) \quad M(u, v) = \frac{1}{\Delta(x)} \prod_{j=1}^n (1 + \frac{v}{x_j} (1 - uT_{t,x_j})) \Delta(x),$$

$$N(u, v) = \frac{1}{\Delta(x)} \prod_{j=1}^n (T_{t,x_j} + \frac{v}{x_j} (1 - uT_{t,x_j})) \Delta(x).$$

Furthermore, their action on Schur functions are computed as follows:

$$(5.7) \quad M(u, v) s_\lambda(x) = \sum_{K \subset [1, n]} v^{|K|} s_{\lambda - (1^K)}(x) \prod_{k \in K} (1 - ut^{\lambda_k + \delta_k}),$$

$$N(u, v) s_\lambda(x) = \sum_{K \subset [1, n]} v^{|K|} s_{\lambda - (1^K)}(x) \prod_{k \in K} (1 - ut^{\lambda_k + \delta_k}) \prod_{k \notin K} t^{\lambda_k + \delta_k}.$$

If $\ell(\lambda) \leq m$ and $u = 1$, we have

$$(5.8) \quad M_m(1) s_\lambda(x) = s_{\lambda - (1^m)}(x) \prod_{k=1}^m (1 - t^{\lambda_k + n - k}),$$

$$N_m(1) s_\lambda(x) = s_{\lambda - (1^m)}(x) \prod_{k=1}^m (1 - t^{\lambda_k + n - k}) t^{\binom{n-m}{2}}.$$

This implies that $M_m^+ = M_m(1)$ and $M_m^- = t^{-\binom{n-m}{2}} N_m(1)$ satisfy the equality (5.2) for the case $q = t$. As we already explained, it completes the proof of Theorem 5.1.

Proposition 5.2. *For each $m = 0, 1, \dots, n$, we have*

$$(5.9) \quad M_m^+ = \frac{1}{\Delta(x)} \sum_{w \in W} \epsilon(w) w \left(x_1^{\delta_1} \cdots x_n^{\delta_n} e_m(\Xi_1, \dots, \Xi_n) \right)$$

where

$$(5.10) \quad \Xi_i = \frac{1}{x_i} (1 - t^{n-i} T_{q, x_i}) \quad (i = 1, \dots, n).$$

Similarly we have

$$(5.11) \quad M_m^- = \frac{t^{\binom{n}{2} - \binom{n-m}{2}}}{\Delta(x)} \sum_{w \in W} \epsilon(w) w \left(x_1^{\delta_1} \cdots x_n^{\delta_n} e_m(\Xi'_1, \dots, \Xi'_n) \right) T_{q, x_1} \cdots T_{q, x_n}$$

where $\Xi'_i = \Xi_i t^{-\delta_i} T_{q, x_i}^{-1}$ ($i = 1, \dots, n$).

From these expressions, we obtain the following quasi-classical limit.

$$(5.12) \quad \mathcal{M}_m = \frac{1}{\Delta(x)} \sum_{w \in W} \epsilon(w) w \left(x_1^{\delta_1} \cdots x_n^{\delta_n} e_m(\mathcal{D}_1, \dots, \mathcal{D}_n) \right)$$

where

$$(5.13) \quad \mathcal{D}_i = \frac{1}{x_i} (\alpha x_i \frac{\partial}{\partial x_i} + n - i) \quad (i = 1, \dots, n).$$

These differential operators are the lowering operators for Jack polynomials:

$$(5.14) \quad \mathcal{M}_m J_\lambda^{(\alpha)}(x) = \prod_{i=1}^m (\alpha \lambda_i + m - i)(\alpha(\lambda_i - 1) + n - i + 1) J_{\lambda - (1^m)}^{(\alpha)}(x),$$

for any partition λ with $\ell(\lambda) \leq m$.

Remark 5.3. It would be an interesting problem to describe the algebra generated by the raising and lowering operators K_m^\pm, M_m^\pm ($m = 0, 1, \dots, n$ or ∞). Another interesting problem is to construct the analogues of raising and lowering operators for Koornwinder's multivariable Askey-Wilson polynomials, as well as for their nonsymmetric versions. We are currently studying these problems and hope to report on them in the near future.

REFERENCES

- [KN] A.N. Kirillov and M. Noumi, *Affine Hecke algebras and raising operators for Macdonald polynomials*, preprint (February 1996).
- [Ma] I.G. Macdonald, *Symmetric Functions and Hall Polynomials* (Second Edition), Oxford Mathematical Monographs, Oxford University Press Inc., New York, 1995.
- [MN] K. Mimachi and M. Noumi, *Notes on eigenfunctions for Macdonald's q -difference operators*, preprint (January, 1996).
- [LV1] L. Lapointe and L. Vinet, *A Rodrigues formula for the Jack polynomials and the Macdonald-Stanley conjecture*, preprint CRM-2294 (1995).
- [LV2] L. Lapointe and L. Vinet, *Exact operator solution of the Calogero-Sutherland model*, preprint CRM-2272 (1995).

A.N. KIRILLOV:

Department of Mathematical Sciences, University of Tokyo,
Komaba, Meguro, Tokyo 153, Japan;
Steklov Mathematical Institute,
Fontanka 27, St. Petersburg 191011, Russia

M. NOUMI:

Department of Mathematics, Faculty of Science, Kobe University,
Rokko, Kobe 657, Japan

E-mail address: kirillov@res.s.u-tokyo.ac.jp, noumi@math.s.kobe-u.ac.jp